Finite temperature amplitudes and reaction rates in Thermofield dynamics.

A. Rakhimov * and F.C. Khanna †

Physics Department, University of Alberta Edmonton, Canada T6G2J1

and

TRIUMF, 4004 Wesbrook Mall, Vancouver, British Columbia, Canada, V6T2A3

Abstract

We propose a method for calculating the reaction rates and transition amplitudes of generic process taking place in a many body system in equilibrium. The relationship of the scattering and decay amplitudes as calculated in Thermo Field Dynamics the conventional techniques is established. It is shown that in many cases the calculations are relatively easy in TFD.

PACS number(s):11.10.Wx,11.15Bt,13.85Fb

Typeset using REVTEX

*Permanant address:Institute of Nuclear Physics, Tashkent, Uzbekistan (CIS)

†E-mail : khanna@phys.ualberta.ca

I. INTRODUCTION

Recent experimental developments in heavy ion collision experiments have lead to the formation of hadronic matter at finite temperature with high densities. At collisions with relativistic heavy ions at facilities like RHIC at Brookhaven and beams at SPS, and eventually at LHC, lead to the deconfinement of quarks from the confined state in hadrons forming a quark gluon plasma. The phase transition is anticipated at temperature of $T_c \approx 150$MeV - 250 MeV. The quark gluon plasma above the critical temperature (T_c) and the hot and dense hadronic matter below T_c possess special problems to investigate their properties at finite temperature and high densities. We have to consider interaction of hadrons, the decay of mesons and neutrons and finally the behavior of quark - quark interactions and the screening effect in the quark gluon plasma (QGP). Such a study requires the development of a formalism of quantum field theory to deal with problems at finite temperature and high densities.

It should be stressed that, developments to deal with hot and dense nuclear matter and hot QGP would provide valuable tools to study early universe as well as the interior of stars. Of course there is a range of problems that would require time dependent treatment since the system may be in a non - equilibrium state that is changing rapidly with time. In general, therefore, it appears that both early Universe and relativistic heavy ion reactions would require a formalism that depends on time and temperature simultaneously.

We wish to explore methods to calculate the reaction rates and decay rates in a medium at temperature T using the real time finite temperature field theory - Thermo Field Dynamics (TFD) [1]. This theory was developed so that the time degree of freedom is not lost [2]. It was realized that, an extention of the usual field theory at zero temperature to finite temperatures would require doubling of the Hilbert space. At the same time the operators have to be doubled such that the new set of operators, designated tilde operators, act on the second Hilbert space, tilde space. Later on it was interpreted that, the second Hilbert space acts like a heat bath that ensures the dynamical system to stay at constant temperature T. It is established that with TFD the Wick's theorem can be applied and other features like Ward - Takahshsi relations and Nambu - Goldstone theorem on spontaneously breaking symmetry can be established. Furthermore as in the case of the usual field theory at zero temperature, Feynman rules for scattering amplitude and decay rates can be established. Therefore the combination of all these feature in TFD allows us to carry out calculations as in the case of $T = 0$ field theory. Some of the essential aspects such as a definition of the vacuum and a definition of the creation and annihilation operators at finite temperature are given in the Appendix. It should be mentioned that for systems in equilibrium there are two other methods available. The first is the imaginary time method due to Matsubara [3] that has been extended to quantum field theory by Jackiw and Dolan [4]. The second method is the closed time path method due to Schwinger and Keldysh [5]. Both these methods are essentually based on Green's function approach. Recently Niegawa [6] has developed a technique to use the closed time method to calculate the scattering amplitude and decay rates. We'll compare this method with that of TFD later on.

It should be stressed that what has been done is to write down the decay amplitudes by using the method of Cutkosky to find the imaginary part of an amplitude. These rules are very helpful in our investigation.

The paper is organised as follows. In sect. II we briefly present basic TFD formulas , which will be nessesary in further calculations. In sect. III we describe our basic formalism. In sections IV,V and VI we consider decay and scattering prococces respectively. In sect VII we consider loop contributions. The summary and conclusions are given in sect VIII.

II. THERMAL FIELDS AND S - MATRIX IN TFD.

The Thermo Field Dynamics is a real time operator formalism of quantum field theory at finite temperature. Any physical state can be constructed from a temperture dependant vacuum $|0(\beta)\rangle$ which is a pure state. The main feature of TFD is that, the thermal average of any operator A is equal to its temperature dependent vacuum expectation value with the : vacuum, $|0(\beta)\rangle$, being obtained from the usual vacuum by a Bogoliubov transformation. Therefore, we have

$$< A >= \langle 0(\beta)|A|0(\beta)\rangle \tag{2.1}$$

where $\beta \equiv 1/k_B T$ with k_B being the Boltzmann constant. Doubling of degrees of freedom of all fields is required. This is achieved through a "tilde" operation: to each zero temperature field $\phi(x)$ a doublet of fields $(\phi(x), \tilde{\phi}(x))$ is attached, the dynamics of which is controlled by a thermal Lagrangian

$$\hat{L} = L(\phi) - \tilde{L}(\phi) = L(\phi) - L^*(\tilde{\phi}) \tag{2.2}$$

The temperature dependent vacuum state $|0(\beta)\rangle$ is annihilated by the temperature dependent physical annihilation operators

$$\begin{aligned} a_k(\beta)|0(\beta)\rangle = c_k(\beta)|0(\beta)\rangle = d_k(\beta)|0(\beta)\rangle = 0 \\ \tilde{a}_k(\beta)|0(\beta)\rangle = \tilde{c}_k(\beta)|0(\beta)\rangle = \tilde{d}_k(\beta)|0(\beta)\rangle = 0 \end{aligned} \tag{2.3}$$

which are obtained through Bogoluibov transformation from usual annihilation, a_k, and creation operators, $a_k^\dagger$. For a Bose system

$$\begin{aligned} a_k &= \cosh\theta_k a_k(\beta) + \sinh\theta_k \tilde{a}_k^\dagger(\beta), \quad & a_k^\dagger &= \cosh\theta_k a_k^\dagger(\beta) + \sinh\theta_k \tilde{a}_k(\beta), \\ \tilde{a}_k^\dagger &= \sinh\theta_k a_k(\beta) + \cosh\theta_k \tilde{a}_k^\dagger(\beta), \quad & \tilde{a}_k &= \sinh\theta_k a_k^\dagger(\beta) + \cosh\theta_k \tilde{a}_k(\beta), \end{aligned} \tag{2.4}$$

where

$$\sinh^2\theta_k = n_B(k) = 1/(e^{\beta|k_0|} - 1), \quad \cosh\theta_k = e^{\beta k_0/2}\sinh\theta_k. \tag{2.5}$$

with k_0 being the energy associated with the four - vector, k. The creation and annihilation operators satisfy ordinary commutation relations

$$[a_k, a_p^\dagger]_- = [\tilde{a}_k, \tilde{a}_p^\dagger]_- = [a_k(\beta), a_p^\dagger(\beta)]_- = [\tilde{a}_k(\beta), \tilde{a}_p^\dagger(\beta)]_- = \delta(\vec{k} - \vec{p}) \tag{2.6}$$

and all other commutators vanish.

For fermions, with the creation, $c_{p,s}^\dagger$ and $d_{p,s}^\dagger$, and annihilation , $c_{p,s}$ and $d_{p,s}$ operators the Bogoliubov transformations lead to the relations

$$\begin{aligned} c_{p,s} &= \cos\theta_{+p} c_{p,s}(\beta) + i\sin\theta_{+p}\tilde{c}_{p,s}^\dagger(\beta), \quad & d_{p,s} &= \cos\theta_{-p} d_{p,s}(\beta) + i\sin\theta_{-p}\tilde{d}_{p,s}^\dagger(\beta), \\ \tilde{c}_{p,s}^\dagger &= i\sin\theta_{+p} c_{p,s}(\beta) + \cos\theta_{+p}\tilde{c}_{p,s}^\dagger(\beta), \quad & \tilde{d}_{p,s}^\dagger &= i\sin\theta_{-p} d_{p,s}(\beta) + \cos\theta_{-p}\tilde{d}_{p,s}^\dagger(\beta), \\ c_{p,s}^\dagger &= \cos\theta_{+p} c_{p,s}^\dagger(\beta) - i\sin\theta_{+p}\tilde{c}_{p,s}(\beta), \quad & d_{p,s}^\dagger &= \cos\theta_{-p} d_{p,s}^\dagger(\beta) - i\sin\theta_{-p}\tilde{d}_{p,s}(\beta), \\ \tilde{c}_{p,s} &= -i\sin\theta_{+p} c_{p,s}^\dagger(\beta) + \cos\theta_{+p}\tilde{c}_{p,s}(\beta), \quad & \tilde{d}_{p,s} &= -i\sin\theta_{-p} d_{p,s}^\dagger(\beta) + \cos\theta_{-p}\tilde{d}_{p,s}(\beta), \end{aligned} \tag{2.7}$$

where

$$\sin^2\theta_{\pm p} = 1/(1+e^{\beta|p_0 \mp \mu|}), \quad \cos\theta_{\pm p} = e^{\beta(p_0 \pm \mu)/2}\sin\theta_{\pm p}$$
$$\sin^2\theta_{+p} = n_F(p), \quad \sin^2\theta_{-p} = \bar{n}_F(p). \tag{2.8}$$

Here p_0 is the energy associated with the four - vector, p and μ is the chemical potential. The anticommutation relations for creation and annihilation operators (c- for particles, and d - for antiparticles) are similar to those at zero temperature:

$$[c_{p's'}c^\dagger_{p,s}]_+ = [d_{p's'}d^\dagger_{p,s}]_+ = [c_{p's'}(\beta)c^\dagger_{p,s}(\beta)]_+ = [d_{p's'}(\beta)d^\dagger_{p,s}(\beta)]_+ = \delta(\vec{p}-\vec{p'})\delta_{ss'}$$
$$[\tilde{c}_{p's'}\tilde{c}^\dagger_{p,s}]_+ = [\tilde{d}_{p's'}\tilde{d}^\dagger_{p,s}]_+ = [\tilde{c}_{p's'}(\beta)\tilde{c}^\dagger_{p,s}(\beta)]_+ = [\tilde{d}_{p's'}(\beta)\tilde{d}^\dagger_{p,s}(\beta)]_+ = \delta(\vec{p}-\vec{p'})\delta_{ss'} \tag{2.9}$$

All other anti - commutators are zero. In order to define the reaction rates and decay widths , we follow the prescription due to Feynman [7], which is best stated by Dyson [8], and this involves writing down an operator:

$$H_F(x_0) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty} dx_1 dx_2 \ldots dx_n T[H^e(x_0)H_I(x_1)H_I(x_2)\ldots H_I(x_n)], \tag{2.10}$$

Then choosing $H^e(x_0) = 1$ this reduces to an expression that is evaluated between an initial and a final state. Therefore we get an expression that looks like and is equal to an S matrix

$$S = \sum_{n=0}^{\infty} S^n = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int dx_1 dx_2 \ldots dx_n T[H_I(x_1)H_I(x_2)\ldots H_I(x_n)], \tag{2.11}$$

Similarly for the $\tilde{H}_I(x_0)$ we get

$$\tilde{S} = \sum_{n=0}^{\infty} \tilde{S}^n = \sum_{n=0}^{\infty}\frac{(+i)^n}{n!}\int dx_1 dx_2 \ldots dx_n T[\tilde{H}_I(x_1)\tilde{H}_I(x_2)\ldots \tilde{H}_I(x_n)], \tag{2.12}$$

To get the total transition we have:

$$\hat{S} = S * \tilde{S} \tag{2.13}$$

Particularly, at the tree level

$$\hat{S} \approx 1 - i\int d^4x(H_I(x) - \tilde{H}_I(x)) \tag{2.14}$$

Now in these expressions $H_I(x)$ and $\tilde{H}_I(x)$ may be introduced explicitly and then Wick's theorem is used as in the case of $T = 0$ field theory to obtain all the contributions in perturbation theory. The evaluation proceeds by using a set of Feynman rules for evaluation of a reaction rate and decay widths. It is clear that for operators appearing in $H_I(x)$ and $\tilde{H}_I(x)$, the transformation of boson and Fermion operators given by Eq.s (2.4) and (2.7) respectively has to be used to bring in the temperature dependend factors. The one particle Green's functions are given explicitly in the Appendix.

III. CALCULATION OF REACTION RATES

Let us consider the process

$$p_1 + p_2 + \ldots + p_n \to p'_1 + p'_2 + \ldots + p'_m \tag{3.1}$$

At zero temperatute, $T = 0$, the amplitude of this process can be calculated by usual Feynman rules:

$$\langle f|\hat{S}|i\rangle = \sum_{n=0}^{\infty} \langle f|\hat{S}^{(n)}|i\rangle \tag{3.2}$$

where $|i\rangle = a^\dagger_{p_1} a^\dagger_{p_2} \ldots a^\dagger_{p_n}|0\rangle$, $|f\rangle = a^\dagger_{p'_1} a^\dagger_{p'_2} \ldots a^\dagger_{p'_m}|0\rangle$ and $|0\rangle$ is an ordinary vacuum state: $a_p|0\rangle = 0$. In the present article we propose that the amplitude and, hence the cross section, of process (3.1) can be directly calculated perturbatively using Eq.s (2.11) - (2.13) at $T \neq$ also. Namely, we propose that

1)The transition amplitude can be determined by Eq. (2.11) - (2.13) where for the process (3.1) we have $|i\rangle = a^\dagger_{p_1}(\beta) a^\dagger_{p_2}(\beta) \ldots a^\dagger_{p_n}(\beta)|0(\beta)\rangle$ and $|f\rangle = a^\dagger_{p'_1}(\beta) a^\dagger_{p'_2}(\beta) \ldots a^\dagger_{p'_m}(\beta)|0(\beta)\rangle$.

2)The phase space factors, relating an amplitude to the rate are the same as in zero temperature case. For example , the differential cross section for the process $p_1 + p_2 \to p'_1 + p'_2 + \ldots + p'_m$ is [9]

$$d\sigma = (2\pi)^4 \delta^4(p'_1 + p'_2 + \ldots + p'_m - p_1 - p_2) \frac{1}{4E_1 E_2 v_{\text{rel}}} \prod_l (2m_l) \prod_{i=1}^{m} \frac{d\vec{p'}_i}{(2\pi)^3 2E'_i} |\mathcal{M}_{\text{fi}}|^2 \tag{3.3}$$

where $E_p = \sqrt{m_i^2 + (\vec{p})^2}$ and v_{rel} is the relative velocity of the two initial particles with momenta p_1 and p_2 . The amplitude is related to $\hat{S}$ matrix in manner similar to the procedure carried out at zero temperature

$$\langle f|\hat{S}|i\rangle = i(2\pi)^4\delta^4(P_f - P_i)\mathcal{M}_{\rm fi}\prod_{ext}[\frac{m}{VE}]^{1/2}\prod_{ext}[\frac{m}{V\omega}]^{1/2}. \tag{3.4}$$

Here P_i and P_f are the total four momenta in the initial and final states, and the products extend over all external fermions and bosons, E and ω being the energies of the individual external fermions and bosons respectively, and V is the volume. We propose that, in TFD this definition still holds.

IV. DECAY OF PARTICLES

We know that at $T = 0$ the decay rates Γ can be calculated either by studying discontinuities of the self energy of a decaying particle (Cutkosky rules [10]) or by direct calculation of the amplitude perturbatively (Feynman rules). Obviously these two methods must lead to the same result.

For finite temperature the Cutkosky rules were generalized by Kobes and Semenoff [11]. In the next section we shall calculate the decay of a particle at finite temperature
a)by using generalized Cutkosky rules (GCR); and also
b) directly as a square of module of elements of $\hat{S}$ matrix, i.e. by using the above method.
It will be established that, these two methods give the same result for Γ. We shall present these to establish their equivalence.

A. $\sigma(k) \to \pi^0(k_1) + \pi^0(k_2)$

Firstly, we calculate the decay rate of sigma meson into two pions (we'll omit isospin indeces for simplicity) in TFD , using GCR. The interaction Lagrangian is $L_{\rm int} = \frac{\lambda}{2}\sigma(x)\pi^2(x)$. The decay rate of a boson with mass M and four momentum $k = (\omega, \vec{k})$ with $\omega = \sqrt{M^2 + (\vec{k})^2}$, is related to the self energy [12] $\bar{\Sigma}(\omega)$ by

$$\Gamma_{GCR}(w) = -\frac{1}{\omega}Im\bar{\Sigma}(\omega) = -\frac{(e^{\beta\omega} - 1)}{\omega(e^{\beta\omega} + 1)}Im\Sigma_{11}(k) \tag{4.1}$$

where

$$i\Sigma_{11}(k) = \lambda^2 \int \frac{dp^4}{(2\pi)^4} i\Delta^0_{11}(p) i\Delta^0_{11}(p-k) \tag{4.2}$$

The GCR [11], as illustrated in Fig. 1 give:

$$2Im\Sigma_{11}(k) = \lambda^2 \int \frac{dp^4}{(2\pi)^4} [i\Delta^+(p) i\Delta^-(p-k) + i\Delta^-(p) i\Delta^+(p-k)] \tag{4.3}$$

where $i\Delta^{\pm}(p) = 2\pi[\Theta(\pm p_0) + n_B(p)]\delta(p^2 - m^2)$. The two terms in the integrand are related by

$$i\Delta^{\pm}(p) = ie^{\pm\beta p_0}\Delta^{\mp}(p) \tag{4.4}$$

From (4.1) -(4.4) we get:

$$\begin{aligned} \Gamma_{GCR}(w) = & \frac{(e^{\beta\omega} - 1)(e^{-\beta\omega} + 1)\lambda^2}{2\omega(e^{\beta\omega} + 1)} \int \frac{dp^4}{(2\pi)^4} (2\pi)^2 \delta(p^2 - m^2)\delta((p-k)^2 - m^2) \times \\ & \times [\Theta(p_0) + n_B(p_0)][\Theta(-p_0 + \omega) + n_B(p_0 - \omega)] \end{aligned} \tag{4.5}$$

We shall proceed and evaluate Γ_{GCR} explicitly. Note , however that, at finite temperature Lorentz invariance is lost, and hence, the decay rate will no longer be Lorentz invariant, but dependent on the reference frame. In the following examples we will choose the rest frame of the decaying particle, as the reference frame, that is we set $\vec{k} = 0$. First, we consider the product of the mass shell δ functions. With $k = (M, 0, 0, 0)$ the compatible zeros are easily found to be $p_0 = \omega_p = M/2$. Hence the delta functions reduce to :

$$\delta(p^2 - m^2)\delta((p-k)^2 - m^2) = \frac{1}{4M\omega_p}\delta(p_0 - \omega_p)\delta(\omega_p - M/2) \tag{4.6}$$

and fix the momentum dependence of the integrand completely.

The thermal factors in (4.5) can be reduced on mass shell: $p_0 = M/2$, $\omega = M$

$$\begin{aligned} & \frac{(e^{\beta\omega} - 1)(e^{-\beta\omega} + 1)}{(e^{\beta\omega} + 1)} [\Theta(p_0) + n_B(p_0)][\Theta(-p_0 + \omega) + n_B(p_0 - \omega)] = \\ & = (1 - e^{-\beta\omega})[1 + n_B(p_0)][1 + n_B(p_0 - \omega)] = (1 - e^{-\beta M})(1 + n_B(M/2))^2 \end{aligned} \tag{4.7}$$

Now inserting (4.6), (4.7) into (4.5) we get:

$$\frac{\Gamma_{GCR}(T \neq 0)}{\Gamma(T = 0)} = \frac{(1 + n_B(M/2))^2}{1 + n_B(M)} = (1 + n_B(M/2))^2 - n_B^2(M/2) \tag{4.8}$$

Here we used the simple relations :

$$
\begin{aligned}
e^{-\beta M} &= e^{-\beta M/2} e^{-\beta M/2} \\
e^{-\beta \omega} &= n_B(\omega)/(1+n_B(\omega))
\end{aligned}
\tag{4.9}
$$

which will be useful for further calculations.

Now we calculate the rate $\Gamma(\sigma \to \pi\pi)$ directly. Because of the doubling of degrees of freedom the whole interaction Lagrangian will be $\hat{L}_{\text{int}} = \lambda\sigma(x)\pi^2(x) - \lambda\tilde{\sigma}(x)\tilde{\pi}^2(x)$. The initial and final states are $|i\rangle = a_k^\dagger(\beta)|0(\beta)\rangle$ and $|f\rangle = b_{k_1}^\dagger(\beta)b_{k_2}^\dagger(\beta)|0(\beta)\rangle$ where operators $a(\beta)$ and $b(\beta)$ stand for σ and π fields respectively. At the tree level (2.14) the matrix element of $\hat{S}$ matrix is given by

$$
\langle f|\hat{S}|i\rangle = i\lambda \int dx \langle 0(\beta)|b_{k_1}(\beta)b_{k_2}(\beta)[\sigma(x)\pi^2(x) - \tilde{\sigma}(x)\tilde{\pi}^2(x)]a_k^\dagger(\beta)|0(\beta)\rangle \tag{4.10}
$$

where the boson fields $\sigma(x)$ and $\pi(x)$ are defined as in eq. (A22). Using Bogoluibov transformations (2.4) and commutation relations (2.6) it is easy to show that:

$$
\begin{aligned}
&\langle 0(\beta)|\sigma(x)a_k^\dagger(\beta)|0(\beta)\rangle = e^{-ikx}\cosh\theta_k, \\
&\langle 0(\beta)|\tilde{\sigma}(x)a_k^\dagger(\beta)|0(\beta)\rangle = e^{-ikx}\sinh\theta_k, \\
&\langle 0(\beta)|b_{k_1}(\beta)b_{k_2}(\beta)\pi^2(x)|0(\beta)\rangle = e^{i(k_1+k_2)x}\cosh\theta_{k_1}\cosh\theta_{k_2} \\
&\langle 0(\beta)|b_{k_1}(\beta)b_{k_2}(\beta)\tilde{\pi}^2(x)|0(\beta)\rangle = e^{i(k_1+k_2)x}\sinh\theta_{k_1}\sinh\theta_{k_2}.
\end{aligned}
\tag{4.11}
$$

From Eqs (4.10),(4.11) and (2.12) we obtain the amplitude:

$$
\mathcal{M}_{\text{fi}}(T) = \lambda[\cosh\theta_k\cosh\theta_{k_1}\cosh\theta_{k_2} - \sinh\theta_k\sinh\theta_{k_1}\sinh\theta_{k_2}]. \tag{4.12}
$$

The decay rate, can be calculated directly from (4.12) without introducing any additional thermal factor:

$$
\begin{aligned}
\Gamma(\omega) &= \frac{1}{2\omega}\int \frac{d\vec{k}_1 d\vec{k}_2 (2\pi)^4 \delta(k-k_1-k_2)|\mathcal{M}_{\text{fi}}(T)|^2}{(2\omega_1)(2\omega_2)(2\pi)^3(2\pi)^3} = \\
&= \frac{\lambda^2}{32\omega\pi^2}\int \frac{d\vec{k}_1 d\vec{k}_2 \delta^4(k-k_1-k_2) W_B(\omega,\omega_1,\omega_2)}{\omega_1\omega_2} \equiv \frac{\lambda^2 I_B(T)}{32\omega\pi^2}
\end{aligned}
\tag{4.13}
$$

where

$$W_B(\omega,\omega_1,\omega_2) = [\cosh\theta_k \cosh\theta_{k_1} \cosh\theta_{k_2} - \sinh\theta_k \sinh\theta_{k_1} \sinh\theta_{k_2}]^2, \tag{4.14}$$

$\omega_i = \sqrt{(\vec{k}_i)^2 + m^2}$ and $\omega = \sqrt{(\vec{k})^2 + M^2}$. Now using the properties of hyperbolic functions and Eqs (2.5), (4.9) this may be futher simplified as follows:

$$\begin{aligned}
&\delta(\omega-\omega_1-\omega_2)W_B(\omega,\omega_1,\omega_2) = n_1 n_2 n_\omega [e^{\beta(\omega+\omega_1+\omega_2)/2} - 1]^2 \delta(\omega-\omega_1-\omega_2) = \\
&= n_1 n_2 n_\omega (e^{\beta\omega} - 1)(e^{\beta(\omega_1+\omega_2} - 1)\delta(\omega-\omega_1-\omega_2) = \\
&= n_1 n_2 n_\omega \{\frac{1+n_\omega}{n_\omega} - 1\}\{\frac{(1+n_1)}{n_1}\frac{(1+n_2)}{n_2} - 1\}\delta(\omega-\omega_1-\omega_2) = \\
&= (1+n_1+n_2)\delta(\omega-\omega_1-\omega_2)
\end{aligned} \tag{4.15}$$

where for simplicity we denote $n_i \equiv n_B(k_i)$, $(i = 1,2)$, and $n_\omega \equiv n_B(k)$.

The integral $I_B(T)$ in Eq. (4.13) can be explicitly calculated in the rest frame of the decaying particle: $\omega = M$, $\vec{k} = 0$, $\omega_i = \sqrt{(\vec{k}_i)^2 + m^2} \equiv \sqrt{(\vec{q})^2 + m^2} = \omega_q$:

$$\begin{aligned}
I_B(T) &\equiv \int \frac{d\vec{k}_1 d\vec{k}_2 \delta^4(k-k_1-k_2)W_B(\omega,\omega_1,\omega_2)}{\omega_1\omega_2} = \\
&= 4\pi \int \frac{dq q^2 \delta(2\omega_q - M) W_B(M,\omega_q,\omega_q)}{\omega_q^2} = 8\pi\sqrt{(1/4-(m/M)^2}W_B(M,M/2,M/2)
\end{aligned} \tag{4.16}$$

Using (4.15) and (4.16) in (4.13) we finally obtain:

$$\frac{\Gamma(T\neq 0)}{\Gamma(T=0)} = W_B(M,M/2,M/2) = (1 + 2n_B(M/2)) \tag{4.17}$$

This formulae is the same as obtained by the GCR method eq. (4.8)

B. $H(k) \rightarrow e^-(k_1) + e^+(k_2)$

As a second example we consider the decay of a boson into two fermions, namely, the decay of Higgs boson into electron - positron pair. The decay rate of this process has been calculated in detail using GCR by Keil [13].The final tree level result is :

$$\frac{\Gamma_{GCR}(T\neq 0)}{\Gamma(T=0)} = \frac{(1-n_F(M/2))(1-\bar{n}_F(M/2))}{1+n_B(M)} \tag{4.18}$$

which is obtained in the rest frame of H boson with a mass M.

Now we evaluate the same rate by direct calculation of the amplitude. The interaction Lagrangian is

$$\hat{L}_{\text{int}} = L_{\text{int}} - \tilde{L}_{\text{int}}, \quad L_{\text{int}} = -igH(x)\bar{\psi}(x)\psi(x), \quad \tilde{L}_{\text{int}} = +ig\tilde{H}(x)\bar{\tilde{\psi}}(x)\tilde{\psi}(x) \tag{4.19}$$

The matrix element of $\hat{S}$ matrix from an initial state $|i\rangle = a_k^\dagger(\beta)|0(\beta)\rangle$ to the final state $|f\rangle = c_{k_1}^\dagger(\beta)d_{k_2}^\dagger(\beta)|0(\beta)\rangle$ at the tree level is

$$\langle f|\hat{S}|i\rangle = i(-ig)\int dx^4\langle i|[H(x)\bar{\psi}(x)\psi(x) + \tilde{H}(x)\bar{\tilde{\psi}}(x)\tilde{\psi}(x)]|f\rangle \tag{4.20}$$

where we omit the spin indices for simplicity. As in previous example, Eq. (4.11), we have

$$\langle f|H(x)|i\rangle = e^{-ikx}\cosh\theta_k, \quad \langle f|\tilde{H}(x)|i\rangle = e^{-ikx}\sinh\theta_k. \tag{4.21}$$

For the fermions , using (A23) in (4.20) we have

$$\begin{aligned}\langle f|\bar{\psi}(x)\psi(x)|i\rangle = \int d\vec{p}d\vec{p'}N_pN_p'\langle 0(\beta)|c_{k_1}(\beta)d_{k_2}(\beta)[c_p^\dagger\bar{u}(p)e^{ipx} + d_p\bar{v}(p)e^{-ipx}]\times \\ \times[c_{p'}u(p')e^{-ip'x} + d_{p'}^\dagger v(p')e^{ip'x}]|0(\beta)\rangle\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\langle f|\bar{\tilde{\psi}}(x)\tilde{\psi}(x)|i\rangle = \int d\vec{p}d\vec{p'}N_pN_p'\langle 0(\beta)|c_{k_1}(\beta)d_{k_2}(\beta)[\tilde{c}_p^\dagger\bar{\tilde{u}}(p)e^{-ipx} + \tilde{d}_p\bar{\tilde{v}}(p)e^{ipx}]\times \\ \times[\tilde{c}_{p'}\tilde{u}(p')e^{ip'x} + \tilde{d}_{p'}^\dagger\tilde{v}(p')e^{-ip'x}]|0(\beta)\rangle\end{aligned} \tag{4.23}$$

The only "surviving" term in (4.22) is that one which includes

$$\begin{aligned}&\langle 0(\beta)|c_{k_1}(\beta)d_{k_2}(\beta)c_p^\dagger d_{p'}^\dagger|0(\beta)\rangle = \langle 0(\beta)|c_{k_1}(\beta)d_{k_2}(\beta)[\cos\theta_{+p}c_p^\dagger(\beta) - i\sin\theta_{+p}\tilde{c}_p(\beta)] = \\ &\quad = [\cos\theta_{-p'}d_{p'}^\dagger(\beta) - i\sin\theta_{-p'}\tilde{d}_{p'}(\beta)]|0(\beta)\rangle = \langle 0(\beta)|c_{k_1}(\beta)d_{k_2}(\beta)c_p^\dagger(\beta)d_{p'}^\dagger(\beta)|0(\beta)\rangle\times \\ &\times\cos\theta_{+p}\cos\theta_{-p'}\cos\theta_{+k_1}\cos\theta_{-k_2}\delta(\vec{k}_1 - \vec{p})\delta(\vec{k}_2 - \vec{p'})\end{aligned} \tag{4.24}$$

Similarly, the nonzero contribution to (4.23) involves

$$\begin{aligned}&\langle 0(\beta)|c_{k_1}(\beta)d_{k_2}(\beta)\tilde{d}_p\tilde{c}_{p'}|0(\beta)\rangle = \langle 0(\beta)|c_{k_1}(\beta)d_{k_2}(\beta)[\cos\theta_{-p}\tilde{d}_p(\beta) - i\sin\theta_{-p}d_p^\dagger(\beta)]\times \\ &\times[\cos\theta_{+p'}\tilde{c}_{p'}(\beta) - i\sin\theta_{+p'}c_{p'}^\dagger(\beta)]|0(\beta)\rangle = \\ &= -\langle 0(\beta)|c_{k_1}(\beta)d_{k_2}(\beta)c_{p'}^\dagger(\beta)d_p^\dagger(\beta)|0(\beta)\rangle\sin\theta_{-p}\sin\theta_{+p'} \\ &= -\sin\theta_{+k_1}\sin\theta_{-k_2}\delta(\vec{k}_1 - \vec{p'})\delta(\vec{k}_2 - \vec{p})\end{aligned} \tag{4.25}$$

The decay of the process is given by a general formula of the zero temperature field theory:

$$\Gamma(\omega)=\frac{1}{2\omega}\int\frac{d\vec{k}_1 d\vec{k}_2(2\pi)^4\delta^4(k-k_1-k_2)(2m)^2|\mathcal{M}_{\rm fi}(T)|^2}{(2\omega_1)(2\omega_2)(2\pi)^3(2\pi)^3} \tag{4.26}$$

where the transition amplitude is given by Eqs (3.4), and for the present case it has the form

$$\mathcal{M}_{\rm fi}(T)=(-ig)[\cos\theta_{k_1}\cos\theta_{-k_2}\cosh\theta_\omega\bar{u}(k_1)v(k_2)-\sin\theta_{k_1}\sin\theta_{-k_2}\sinh\theta_\omega\bar{\tilde{v}}(k_2)\tilde{u}(k_1)] \tag{4.27}$$

Then Eq. (4.26) can be fiurther simplified by using

$$\sum_{\rm spins}|\mathcal{M}_{\rm fi}(T)|^2=g^2[\cos\theta_{k_1}\cos\theta_{-k_2}\cosh\theta_\omega-\sin\theta_{k_1}\sin\theta_{-k_2}\sinh\theta_\omega]^2 Tr\frac{(\not{k}_1+m)}{2m}\frac{(\not{k}_2+m)}{2m} \tag{4.28}$$

and

$$\begin{aligned}&[\cos\theta_{k_1}\cos\theta_{-k_2}\cosh\theta_\omega-\sin\theta_{k_1}\sin\theta_{-k_2}\sinh\theta_\omega]^2=\\&=\cos^2\theta_{k_1}\cos^2\theta_{-k_2}\cosh^2\theta_\omega(1-e^{-\beta(\omega+\omega_1+\omega_2)/2})^2\delta(\omega-\omega_1-\omega_2)=\\&=(1-n_F(k_1)(1-\bar{n}_F(k_2)(1-e^{-\beta\omega})\delta(\omega-\omega_1-\omega_2)\equiv\delta(\omega-\omega_1-\omega_2)W_F(\omega,\omega_1,\omega_2)\end{aligned} \tag{4.29}$$

Now, inserting (4.27) -(4.29) into (4.26) and performing integration in the rest frame of the decaying boson we obtain

$$\frac{\Gamma(T\neq 0)}{\Gamma(T=0)}=W_F(M,M/2,M/2)=(1-n_F(M/2))(1-\bar{n}_F(M/2))(1-e^{-\beta M}) \tag{4.30}$$

which is exactly the same as Eq. (4.18) given by GCR, since $(1-e^{-\beta\omega})=(\cosh^2\theta_\omega)^{-1}=(1+n_B(\omega))^{-1}$

C. $\Phi(k)\rightarrow\phi(k_1)+\phi(k_2)+\phi(k_3)$

The next example is the decay rate of a boson -$\Phi(x)$ into three bosons - $\phi(x)$ with the interaction Lagrangian $L_{\rm int}=-\frac{\lambda}{3!}\Phi(x)\phi^3(x)$, and $\hat{L}_{int}=L_{\rm int}-\tilde{L}_{int}$ . . The decay rate of the process (both in imaginary time formalism and in TFD) using GCR has been obtained by Fujimoto et al. [12] to be

$$\Gamma_{GCR}(w) = -\frac{Im\bar{\Sigma}(\omega)}{\omega} = \pi\lambda^2 \int \frac{d\vec{k}_1 d\vec{k}_2 \delta(\omega-\omega_1-\omega_2-\omega_3)}{(2\pi)^6 8\omega\omega_1\omega_2\omega_3} \times \\ \times \quad [(1+n_1)(1+n_2)(1+n_3) - n_1 n_2 n_3] \tag{4.31}$$

where $n_i = n_B(k_i)$, $\omega_i = \sqrt{(\vec{k}_i)^2 + m^2}$, $(i = 1, 2, 3)$ and $\omega = \sqrt{(\vec{k})^2 + M^2}$. On the other hand the decay rate is related to the transition amplitude as:

$$\Gamma(\omega) = \frac{1}{2\omega} \int \frac{d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 (2\pi)^4 \delta(k-k_1-k_2-k_3) |\mathcal{M}_{\rm fi}(T)|^2}{(2\omega_1)(2\pi)^3(2\omega_2)(2\pi)^3(2\omega_3)(2\pi)^3} \tag{4.32}$$

where the amplitude $\mathcal{M}_{\rm fi}(T)$ is defined by Eq.(3.4) with the elements of $\hat{S}$ - matrix:

$$\langle f|\hat{S}|i\rangle = i\lambda \int dx \langle 0(\beta)| b_{k_1}(\beta) b_{k_2}(\beta) b_{k_3}(\beta) [\Phi(x)\phi^3(x) - \tilde{\Phi}(x)\tilde{\phi}^3(x)] a_k^\dagger(\beta) |0(\beta)\rangle \tag{4.33}$$

As in the previous example, we get

$$\begin{aligned}
&\delta(\omega-\omega_1-\omega_2-\omega_3)|\mathcal{M}_{\rm fi}(T)|^2 = \\
&= [\cosh\theta_k \cosh\theta_{k_1} \cosh\theta_{k_2} \cosh\theta_{k_3} - \sinh\theta_k \sinh\theta_{k_1} \sinh\theta_{k_2} \sinh\theta_{k_3}]^2 \delta(\omega-\omega_1-\omega_2-\omega_3) = \\
&= n_1 n_2 n_3 n_\omega [1 - e^{\beta(\omega+\omega_1+\omega_2+\omega_3)/2}]^2 \delta(\omega-\omega_1-\omega_2-\omega_3) = \\
&= n_1 n_2 n_3 n_\omega [1 - e^{\beta\omega}][1 - e^{\beta(\omega_1+\omega_2+\omega_3)}] \delta(\omega-\omega_1-\omega_2-\omega_3) \\
&= n_1 n_2 n_3 n_\omega [1 - \frac{1+n_\omega}{n_\omega}][1 - \frac{(1+n_1)(1+n_2)(1+n_3)}{n_1 n_2 n_3}] \delta(\omega-\omega_1-\omega_2-\omega_3) = \\
&= [(1+n_1)(1+n_2)(1+n_3) - n_1 n_2 n_3] \delta(\omega-\omega_1-\omega_2-\omega_3)
\end{aligned} \tag{4.34}$$

where $n_\omega = n_B(k)$, $n_i = n_B(k_i)$. Then, by using (4.34) in (4.32) the result given in Eq. (4.31) is reproduced. Again the direct calculations and those from GCR agree.

V. THE DETAILED BALANCE.

The last example of the previous section can be easily generalized for the decay of a boson into m particles: $\Phi(k) \to \phi(k_1) + \phi(k_2) + \ldots + \phi(k_m)$ with the decay rate

$$\Gamma(\omega) = \frac{1}{2\omega}\int \frac{d\vec{k}_1 d\vec{k}_2 \ldots d\vec{k}_m (2\pi)^4 \delta^4(k - k_1 - k_2 - \ldots - k_m)|\mathcal{M}_{\mathrm{fi}}(T)|^2}{(2\omega_1)(2\omega_2)\ldots(2\omega_m)(2\pi)^{3m}} \tag{5.1}$$

where

$$\begin{aligned} &\delta(\omega - \omega_1 - \ldots - \omega_m)|\mathcal{M}_{\mathrm{fi}}(T)|^2 = \delta(\omega - \omega_1 - \ldots - \omega_m) \\ &[\cosh\theta_k \cosh\theta_{k_1} \ldots \cosh\theta_{k_m} - \sinh\theta_k \sinh\theta_{k_1} \ldots \sinh\theta_{k_m}]^2 = \\ &= n_1 \ldots n_m n_\omega [1 - e^{\beta\omega}][1 - e^{\beta(\omega_1 + \ldots + \omega_m)}] = \\ &= [(1 + n_1)(1 + n_2)\ldots(1 + n_m) - n_1 n_2 \ldots n_m]\delta(\omega - \omega_1 - \ldots - \omega_m) \end{aligned} \tag{5.2}$$

It is clear from Eq. (5.1) and (5.2) that, the total decay rate $\Gamma(\omega)$ is the difference of the forward rate of the boson decay - $\Gamma_d(\omega)$ and the rate of the inverse process $\Gamma_i(\omega)$: $\Gamma(\omega) = \Gamma_d(\omega) - \Gamma_i(\omega)$, which are related by the principle of detailed balance:

$$\frac{\Gamma_d(\omega)}{\Gamma_i(\omega)} = \frac{(1 + n_1)(1 + n_2)\ldots(1 + n_m)}{n_1 n_2 \ldots n_m} = e^{\beta(\omega_1 + \omega_2 + \ldots + \omega_m)} = e^{\beta\omega} \tag{5.3}$$

This relation was first shown by Weldon, [14] by analysing the discontiniuties of the self energy function in Matsubara formalism. Thus, we conclude that, our approach of direct calculation of rates through elements of the $\hat{S}$ matrix leads to the correct relation for the detailed balance principle at finite temperature. Note that, the Eqs. (5.1) - (5.3) reveal the the importance of doubling of degrees of freedom at nonzero temperature: If we had neglected the term $\tilde{L}_{\mathrm{int}}$ then the term proportional to "$\sinh\theta_k$" in Eq. (5.2) would have vanished giving a wrong relation. Although the "tilde" particles are introduced as an essential part of the finite temperature formalism as fictitious particles, they play an important role in direct and inverse processes taking place at finite temperature. It is important to remark that there are no transitions , and hence no matrix element, between the tilde and non - tilde particles.

However, following Keil [13], we underline that, $\Gamma(\omega)$ and not the partial rates $\Gamma_d(\omega)$ and $\Gamma_i(\omega)$ represent the physically measurable decay rate! The reason is that, it is the full decay width $\Gamma(\omega) = \Gamma_d(\omega) - \Gamma_i(\omega)$ that is connected to the pole of the Green's function [12,15]

$$\begin{aligned} \Delta_{11}(k) &= \frac{1 + n_B(k_0)}{k^2 - m_0^2 - \bar{\Sigma} + i\varepsilon} - \frac{n_B(k_0)}{k^2 - m_0^2 - \bar{\Sigma}^* - i\varepsilon} = \\ &\frac{1 + n_B(k_0)}{(k_0 + i\Gamma/2)^2 - \omega^2 + i\varepsilon} - \frac{n_B(k_0)}{(k_0 - i\Gamma/2)^2 - \omega^2 - i\varepsilon} \end{aligned} \tag{5.4}$$

where $\omega^2 = k_0^2 + Re\bar{\Sigma} - \Gamma^2/4$, $\Gamma = -Im\bar{\Sigma}/k_0$, $k_0 = \sqrt{(\vec{k})^2 + m^2}$.

VI. SCATTERING CROSS SECTION OF $1 + 2 \rightarrow 1' + 2'$.

As it was stated in Sect III in our approach, in accordance with the concept of TFD, the relation between the rate (cross section) and the transition amplitude is the same as it is in the zero temperature quantum field theory. For example the cross section of the process $a(k_1) + b(k_2) \rightarrow a'(k_1{}') + b'(k_2{}')$ is [9]

$$\frac{d\sigma}{d\Omega'} = \frac{|\mathcal{M}_{\mathrm{fi}}(T)|^2(\prod_l(2m_l))|\vec{k_1}'|^2}{64\pi^2 v_{\mathrm{rel}}\omega_1\omega_2\omega_1'\omega_2'}\{\frac{\partial(\omega_1' + \omega_2')}{\partial|\vec{k_1}'|}\}^{-1} \tag{6.1}$$

in the usual notation. In particular , the cross section of the elastic scattering of two bosons in their center of mass (CoM) frame ($\vec{k_1}' = -\vec{k_2}'$) is

$$(\frac{d\sigma}{d\Omega}) = \frac{|\mathcal{M}_{\mathrm{fi}}(T)|^2}{64\pi^2(\omega_1 + \omega_2)^2} \tag{6.2}$$

These equations imply that

$$\frac{(d\sigma/d\Omega')|_{T\neq 0}}{(d\sigma/d\Omega')|_{T=0}} = \frac{|\mathcal{M}_{\mathrm{fi}}(T \neq 0)|^2}{|\mathcal{M}_{\mathrm{fi}}(T = 0)|^2} \equiv W(T) \tag{6.3}$$

where the amplitudes may be determined through Eq.s (3.2)-(3.4). To see the consequences of this relation we will consider some examples.

A. Boson - Boson scattering Let us assume that both a and b particles are bosons with the interaction Lagrangian $L_{\mathrm{int}} = \lambda\phi_a^2(x)\phi_b^2(x)$ and $\hat{L}_{int} = L_{\mathrm{int}} - \tilde{L}_{int}$. At the tree level , using the results given in Eq.s (3.2) - (3.4) and Eq. (4.11) in (6.3) we get

$$\begin{aligned} W_{BB}(T) = \frac{|\mathcal{M}_{\mathrm{fi}}(T \neq 0)|^2}{|\mathcal{M}_{\mathrm{fi}}(T = 0)|^2} = [C(T) - S(T)]^2 = \\ = n_1 n_2 n_1' n_2'[e^{\beta(\omega_1 + \omega_2 + \omega_1' + \omega_2')/2)} - 1]^2, \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} C(T) = \cosh\theta_{k_1}\cosh\theta_{k_2}\cosh\theta_{k_1'}\cosh\theta_{k_2'}, \\ S(T) = \sinh\theta_{k_1}\sinh\theta_{k_2}\sinh\theta_{k_1'}\sinh\theta_{k_2'}, \end{aligned} \tag{6.5}$$

and $n_i = n_B(k_i)$, $n_i' = n_B(k_i')$. Due to energy conservation we have

$$[e^{\beta(\omega_1 + \omega_2 + \omega_1' + \omega_2')/2} - 1]^2 = (e^{\beta(\omega_1 + \omega_2)} - 1)(e^{\beta(\omega_1' + \omega_2')} - 1) \tag{6.6}$$

Now we may express the exponents via n_i, n_i'

$$e^{\beta\omega_i} = \frac{1 + n_i}{n_i}, \qquad e^{\beta\omega_i'} = \frac{1 + n_i'}{n_i'}; \tag{6.7}$$

to get

$$\begin{aligned} W_{BB}(T) &= (1 + n_1 + n_2)(1 + n_1' + n_2') = \\ &= [(1 + n_1)(1 + n_2) - n_1 n_2][(1 + n_1')(1 + n_2') - n_1' n_2']. \end{aligned} \tag{6.8}$$

B. Fermion - Fermion scattering

Now let us assume that both a and b particles are fermions with the interaction lagrangian $L_{\text{int}} = g\bar{\psi}_a(x)\Gamma_\alpha\psi_a(x)\bar{\psi}_b(x)\Gamma^\alpha\psi_b(x)$ where Γ_α is an appropriate combination of Dirac matrices. Similarly as in the previous case we get :

$$\begin{aligned} W_{FF}(T) = \frac{|\mathcal{M}_{\text{fi}}(T \neq 0)|^2}{|\mathcal{M}_{\text{fi}}(T = 0)|^2} = [\cos\theta_{k_1}\cos\theta_{k_2}\cos\theta_{k_1'}\cos\theta_{k_2'} - \sin\theta_{k_1}\sin\theta_{k_2}\sin\theta_{k_1'}\sin\theta_{k_2'}]^2 = \\ n_F(k_1)n_F(k_2)n_F(k_1')n_F(k_2')[e^{\beta(\omega_1 + \omega_2 + \omega_1' + \omega_2' - \mu_1 - \mu_2 - \mu_1' - \mu_2')/2} - 1]^2. \end{aligned} \tag{6.9}$$

At equilibrum $\mu_1 + \mu_2 = \mu_1' + \mu_2'$, and hence, by using

$$\begin{aligned} [e^{\beta(\omega_1 - \mu_1 + \omega_2 - \mu_2)} - 1]^2 = [e^{\beta(\omega_1 - \mu_1 + \omega_2 - \mu_2)} - 1][e^{\beta(\omega_1' - \mu_1' + \omega_2' - \mu_2')} - 1] = \\ = [\frac{1}{n_F(k_1)n_F(k_2)} - \frac{1}{n_F(k_1)} - \frac{1}{n_F(k_2)}][\frac{1}{n_F(k_1')n_F(k_2')} - \frac{1}{n_F(k_1')} - \frac{1}{n_F(k_2')}], \end{aligned} \tag{6.10}$$

we get

$$W_{FF}(T) = \frac{|\mathcal{M}_{\text{fi}}(T \neq 0)|^2}{|\mathcal{M}_{\text{fi}}(T = 0)|^2} = [(1 - n_1)(1 - n_2) - n_1 n_2][(1 - n_1')(1 - n_2') - n_1' n_2']. \tag{6.11}$$

where $n_i \equiv n_F(k_i)$, and $n_i' \equiv n_F(k_i')$.

C. Fermion - Boson scattering

Let us assume that particle a is a fermion and particle b is a boson with the interaction lagrangian given as $L_{\text{int}} = -ig\bar{\psi}(x)\Gamma\psi(x)\phi(x)$ (for example $\gamma = \gamma_5$ for pion - nucleon scattering.) In this case we get at tree level

$$W_{FB}(T) = \frac{|\mathcal{M}_{\text{fi}}(T \neq 0)|^2}{|\mathcal{M}_{\text{fi}}(T = 0)|^2} = [\cos\theta_{k_1}\cosh\theta_{k_2}\cos\theta_{k_1{}'}\cosh\theta_{k_2{}'} + \sin\theta_{k_1}\sinh\theta_{k_2}\sin\theta_{k_1{}'}\sinh\theta_{k_2{}'}]^2 \tag{6.12}$$

Then using similar manipulations, as in previous examples we get

$$W_{FB}(T) = \frac{|\mathcal{M}_{\text{fi}}(T \neq 0)|^2}{|\mathcal{M}_{\text{fi}}(T = 0)|^2} = [1 - n_F(k_1) + n_B(k_2)][1 - n_F(k_1{}') + n_B(k_2{}')]. \tag{6.13}$$

The expressions (6.8), (6.11) and (6.13) for the relation of in medium cross sections of $1 + 2 \to 1' + 2'$ scattering can be interpreted physically in the spirit of Weldon [14]. For example $W_{FB}(T)$ can be rewritten as

$$\begin{aligned}
W_{FB}(T) = & (1 - n_F(k_1))(1 + n_B(k_2))(1 - n_F(k_1{}'))(1 + n_B(k_2{}'))+ \\
& + n_F(k_1)n_B(k_2)n_F(k_1{}')n_B(k_2{}') + n_F(k_1)n_B(k_2)(1 - n_F(k_1{}'))(1 + n_B(k_2{}'))+ \\
& + n_F(k_1{}')n_B(k_2{}')(1 - n_F(k_1))(1 + n_B(k_2))
\end{aligned} \tag{6.14}$$

The first term may be interpreted as the probability for emission of all four particles with statistical weights $(1 - n_F)$ for fermions and $(1 + n_B)$ for bosons; the second term is their absorption with statistical weights n_F and n_B. The third term is a direct scattering of two particles, and the fourth one is its inverse. We see that at $T \neq 0$ all these four processes can take place at least virtually. However, we again stress that, only the appropriate sum of all these transitions, that is the expression given by (6.14) is related to the physically measurable temperature dependence of the in - medium cross section.

VII. LOOP CORRECTIONS

All the above results have been obtained at the tree level. Now we wish to show how the method can be handled beyond the tree level. For simplicity we'll choose the $\phi^4(x)$ interaction :$L_{\text{int}} = -\frac{\lambda}{4!} : \phi^4(x) :$. The $\hat{S}$ matrix up to order λ^2 is given by :

$$\hat{S} = S * \tilde{S} \approx 1 - i\int dx^4 [H_I(x) - \tilde{H}_I(x)] + \\ + \frac{(i)^2}{2!}\int dxdy\{T[H_I(x)H_I(y)] + T[\tilde{H}_I(x)\tilde{H}_I(y)]\} \equiv \hat{S}^{(1)} + \hat{S}^{(2)}. \tag{7.1}$$

For the scattering process from the initial state

$$|i\rangle = |\phi(k_1)\phi(k_2)\rangle = a^\dagger_{k_1}(\beta)a^\dagger_{k_2}(\beta)|0(\beta)\rangle \tag{7.2}$$

to a final state

$$|f\rangle = |\phi(k_1{}')\phi(k_2{}')\rangle = a^\dagger_{k_1{}'}(\beta)a^\dagger_{k_2{}'}(\beta)|0(\beta)\rangle \tag{7.3}$$

the matrix element $\langle f|\hat{S}^{(1)}|i\rangle$ and hence the amplitude $\mathcal{M}_{\text{fi}}(T) = \mathcal{M}^{(1)}_{\text{fi}}(T) + \mathcal{M}^{(2)}_{\text{fi}}(T)$, is defined as,

$$\langle f|\hat{S}^{(1)} + \hat{S}^{(2)}|i\rangle = \delta_{if} + \delta^4(P - P')\frac{\mathcal{M}^{(1)}_{\text{fi}}(T) + \mathcal{M}^{(2)}_{\text{fi}}(T)}{(4\pi)^2\sqrt{\omega_1\omega_2\omega'_1\omega'_2}}, \tag{7.4}$$

with $P = k_1 + k_2$ and $P' = k_1{}' + k_2{}'$. $\mathcal{M}^{(1)}_{\text{fi}}(T)$ and $\mathcal{M}^{(2)}_{\text{fi}}(T)$ can be simply calculated by using the procedure used in sec. VI. For instance,

$$\mathcal{M}^{(1)}_{\text{fi}}(T) = -i\lambda[C(T) - S(T)] \tag{7.5}$$

where $C(T)$ and $S(T)$ are defined by Eqs (6.5).

To evaluate the $\mathcal{M}^{(2)}_{\text{fi}}(T)$ the generalized Wick theorem [16] should be used and 2×2 Green's function $\Delta_{ab}(q)$ (see Appendix) is introduced. In particular:

$$i\Delta^0_{11}(x-y) = \langle 0(\beta)|T[\phi(x)\phi(y)]|0(\beta)\rangle = i\int \frac{dp^4}{(2\pi^4)}e^{-iq(x-y)}\Delta^0_{11}(q) \\ i\Delta^0_{22}(x-y) = \langle 0(\beta)|T[\phi(x)\phi(y)]|0(\beta)\rangle = i\int \frac{dp^4}{(2\pi^4)}e^{-iq(x-y)}\Delta^0_{22}(q) \tag{7.6}$$

In terms of these Green's functions, the amplitude is proportional to λ^2 and is

$$\mathcal{M}^{(2)}_{\text{fi}}(T) = \lambda^2[C(T)\int \frac{dq^4}{(2\pi^4)}\Delta^0_{11}(q)\Delta^0_{11}(P-q) + S(T)\int \frac{dq^4}{(2\pi^4)}\Delta^0_{22}(q)\Delta^0_{22}(P-q)]. \tag{7.7}$$

The two terms in the integrand are related by

$$\Delta^0_{11}(q) = -[\Delta^0_{22}(q)]^* = \frac{1}{q^2 - m^2 + i\varepsilon} - 2i\pi n_B(q)\delta(q^2 - m^2) \tag{7.8}$$

and hence Eq. (7.7) simplifies to

$$\mathcal{M}^{(2)}_{\text{fi}}(T) = \lambda^2[(C(T)F(P,T) + S(T)F^*(P,T)],$$
$$F(P,T) = \int \frac{dq^4}{(2\pi^4)}\Delta^0_{11}(q)\Delta^0_{11}(P-q) = I_1(P) - 4\pi i I_2(P) - 4\pi^2 I_3(P) \tag{7.9}$$

with

$$I_1(P) = \int \frac{dq^4}{(2\pi)^4}\frac{1}{(q^2-m^2+i\varepsilon)((P-q)^2-m^2+i\varepsilon)}$$

$$I_2(P) = \int \frac{dq^4}{(2\pi)^4}\frac{n_B(q)\delta(q^2-m^2)}{(P-q)^2-m^2} \tag{7.10}$$

$$I_3(P) = \int \frac{dq^4}{(2\pi)^4} n_B(q) n_B(P-q)\delta(q^2-m^2)\delta((P-q)^2-m^2)$$

The integrals $I_1(P), I_2(P), I_3(P)$ can be evaluated in the CoM frame of the two particles , where $P^2 = (k^0_1 + k^0_2)^2 = E^2$ - total energy of the system.

The integral $I_1(P)$ is logarithmically divergent and can be treated using a renormalization procedure and this finite contribution is called the vacuum fluctuation correction. This term exists even at zero temperature, and can be calculated by the usual method of dimensional regularization [17]. As a result $I_1(P)$ may be written as:

$$I_1(P) = \frac{iN_\varepsilon}{16\pi^2} + \frac{i}{16\pi^2}[b_0(E) - \ln\frac{m^2}{\mu^2}] - \frac{\sqrt{D_0}}{16\pi}$$
$$b_0(E) = 2 + \sqrt{D_0}\ln|\frac{1-\sqrt{D_0}}{1+\sqrt{D_0}}| \tag{7.11}$$
$$D_0 = 1 - 4m^2/E,$$

where the first term involving $N_\varepsilon = 1/\varepsilon - \gamma_0 + \ln 4\pi$ is infinite. To eliminate the latter one should introduce the following counterterm in the Lagrangian [17]

$$L_{\text{contr}} = -\frac{\lambda^2 N_\varepsilon}{16\pi^2 4!} : \phi^4(x) : \tag{7.12}$$

which clearly does not depend on temperature. On the other hand, in accordance with the tilde oparation rule, this automatically leads us to introduce its tilde partner also:

$$\tilde{L}_{\text{contr}} = -\frac{\lambda^2 N_\varepsilon}{16\pi^2 4!} : \tilde{\phi}^4(x) : \tag{7.13}$$

which serves to compensate the divergence, coming from $S(T) * F^*(P,T)$ term in Eq. (7.9).

The $I_2(P)$ is convergent and can be evaluated numerically:

$$I_2(P) = \int \frac{d\vec{q}}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{dq_0 n_B(q_0)[\delta(q_0+|\omega_q|) - \delta(q_0-|\omega_q|)]}{2\omega_q((P-q)^2-m^2)} =$$

$$= \frac{1}{4\pi^3} \int \frac{dq q^2 n_B(q)}{E^2(1-4\omega_q^2/E^2)\omega_q} \equiv \frac{1}{64\pi^3} \bar{I}_2(E,T) \tag{7.14}$$

The integral I_3 including two $\delta-$ functions may be evaluated in a similar way as in sect.IV (Eq.(4.6)) and is given as

$$I_3(P) = \int \frac{dq_0 d\vec{q} n_B(P-q) n_B(q)}{(2\pi)^4 4E\omega_q} \delta(q_0-\omega_q)\delta(\omega_q - E/2) = \frac{n_B^2(E/2)\sqrt{D_0}}{32\pi^3} \tag{7.15}$$

Now , using Eqs. (7.5), (7.9)-(7.15) and introducing $\overline{\mathcal{M}}_{\rm fi} = \mathcal{M}_{\rm fi}/(-i)$ gives

$$\begin{aligned} &\overline{\mathcal{M}}_{\rm fi}(T) = Re\overline{\mathcal{M}}_{\rm fi}(T) + iIm\overline{\mathcal{M}}_{\rm fi}(T) \\ &Re\overline{\mathcal{M}}_{\rm fi}(T) = \lambda(C(T)-S(T))(1-\lambda A(E,T)/16\pi^2) \\ &Im\overline{\mathcal{M}}_{\rm fi}(T) = -\lambda^2(1+2n_B^2(E/2))\sqrt{D_0}(C(T)+S(T))/(16\pi) \end{aligned} \tag{7.16}$$

with $A(E,T) = b_0(E) - \ln\frac{m^2}{\mu^2} - \bar{I}_2(E,T)$. It is interesting to note that $Im\overline{\mathcal{M}}_{\rm fi}(T)$ coincides with the imaginary part of the diagram, given by GCR [11].

From (7.16) one may get the relation for the in - medium elastic cross section in $\phi^4(x)$ scalar theory up to one loop approximation:

$$\frac{(d\sigma/d\Omega')|_{T\neq 0}}{(d\sigma/d\Omega')|_{T=0}} = \frac{(C(T)-S(T))^2(1-\bar{\lambda}A(E,T))^2 + \pi^2\bar{\lambda}^2 D_0(1+2n_B^2(E/2))^2(C(T)+S(T))^2}{[1-\bar{\lambda}(b_0(E)-\ln\frac{m^2}{\mu^2})]^2 + \pi^2 D_0\bar{\lambda}^2} \tag{7.17}$$

where $\bar{\lambda} = \lambda/16\pi^2$, $C(T) = (1+n_B(E/2))^2$ and $S(T) = n_B^2(E/2)$

The appropriate Feynman rules for ϕ^4 scalar theory at finite temperature can be explicitly obtained from Fig.2. They are similar to those for zero temperature. The main differences are the following.

The amplitude consists of two parts : $\mathcal{M}_{\rm fi}(T)$ and $\widetilde{\mathcal{M}}_{\rm fi}(T)$. To calculate $\mathcal{M}_{\rm fi}(T)$

1)Attach the factor $\cosh\theta_p$ to each external leg (incoming or outgoing particle) with momentum p.

2)Attach $i\Delta_{11}(q)$ to the propagator carrying momentum q.

3)Attach factor $-i\lambda$ to each vertex.

To calculate $\widetilde{\mathcal{M}}_{\rm fi}(T)$

1)Attach the factor $\sinh\theta_p$ to each external leg (incoming or outgoing particle) with momentum p.

2)Attach $i\Delta_{22}(q)$ to the propagator carrying momentum q.

3)Attach factor $+i\lambda$ to each vertex.

Using these rules one may calculate any process in ϕ^4 theory at finite temperature to any order of perturbative theory.

VIII. DISCUSSIONS AND SUMMARY

In the present paper the decay amplitude and scattering process are studied in TFD. We have proved that, in the TFD formalism , the rate and transition amplitude of a process can be evaluated in a similar way as in the zero temperature case. The amplitudes are written by establishing a set of Feynman rules using the Lagrangian $\hat{L} = L - \tilde{L}$. The Feynman diagrams are considered. Use of Bogoluibov transformations to define a pure vacuum state lead to Feynman rules for including temperature dependant factors for a process occuring in a system at finite temperature.It will be interesting to evaluate for example NN scattering in nuclear matter at finite temperature. Temperature dependence of coupling constants and masses have been explored in perturbative theory of TFD [21,22]. At present we are attempting a self consistent calculation of these quantities. The role of Ward Takahashi relations at finite T will be crucial to obtaining a self consistent solution.

The present discussion for nuclear matter in equilibrium at finite T can be easily extended to QGP at temperature above the deconfinement phase transition. Some of the interesting properties like screening length and quark - quark interaction at finite temperature have been investigated at finite T [23]. The results are obtained in a simple and straightforward manner.There are numerous other properties of nuclear matter and QGP that may be calculated and would be helpful in studies of many particle hadronic systems.

The present approach simplifies calculation of concrete processes taking place in hot and dense matter. Examples are neutrino electron scattering in thermal plasma [18,19] , β - decay, which plays an important role in cosmology, in -medium nucleon - nucleon cross sections [20] taking place in ion - ion collisions etc. These processes are being investigated with the present approach.

ACKNOWLEDGMENTS

We thank A.E. Santana, M. Revzen and R. Kobes for useful discussions. A.M. Rakhimov is indebted to the University of Alberta for hospitality during his stay, where the main part of this work was performed. The research of F. Khanna is supported in part by National Science and Engineering Research Council of Canada.

APPENDIX A

A. The nessessity of doubling

In the statistical mechanics the statistical average of a quantity A is given by

$$< A >= Z^{-1}(\beta)\mathrm{Tr}[Ae^{-\beta\mathcal{H}}] \tag{A1}$$

where $\mathcal{H} = H - \mu N$, $Z(\beta) = \mathrm{Tr}[e^{-\beta\mathcal{H}}]$. In the zero temperature field theory the average of an observable A is determined by it's vaccuum expectation value:

$$< A >= \langle 0|A|0\rangle \tag{A2}$$

To make the temperature formalism completely parallel to the zero temperature one we assume that there exists a " thermal vaccuum state", such that

$$< A >= \langle 0(\beta)|A|0(\beta)\rangle = Z^{-1}\sum_n e^{-\beta E_n}\langle n|A|n\rangle \tag{A3}$$

for any operator A. Since the Hilbert state is complete we can expand $|0(\beta)\rangle$ in terms of $|n\rangle$ as

$$|0(\beta)\rangle = \sum_n |n\rangle\langle n|0(\beta)\rangle \equiv \sum_n f_n(\beta)|n\rangle \tag{A4}$$

From the last equation we obtain

$$\langle 0(\beta)|A|0(\beta)\rangle = \sum_{n,m} f_n^*(\beta)f_m(\beta)\langle n|A|m\rangle \tag{A5}$$

which will agree with eq. (A3) provided

$$f_n^*(\beta)f_m(\beta) = Z^{-1}e^{-\beta E_n}\delta_{mn} \tag{A6}$$

However as we see from eq. (A4), $f_n(\beta)$'s are ordinary numbers and therefore it is not possible to satisfy (A6). This shows that as long as we restrict ourselves to the original Hilbert space, we cannot define a finite temperature vacuum which would satisfy eq. (A3).Therefore we have to double the Hilbert space, introducing a fictitious system identical to the original system. Let us denote this auxiliary system as a tilde system. Mathematically this means that

$$|n, \tilde{m}\rangle = |n\rangle \otimes |\tilde{m}\rangle \tag{A7}$$

Now we can expand $|0(\beta)\rangle$ in terms of these $|n, \tilde{m}\rangle$ states

$$|0(\beta)\rangle = \sum_n f_n(\beta)|n, \tilde{n}\rangle = \sum_n f_n(\beta)|n\rangle \otimes |\tilde{n}\rangle \tag{A8}$$

In this case we note that

$$\begin{aligned} \langle 0(\beta)|A|0(\beta)\rangle &= \sum_{n,m} f_n^*(\beta) f_m(\beta) \langle n, \tilde{n}|A|m, \tilde{m}\rangle = \\ &= \sum_{n,m} f_n^*(\beta) f_m(\beta) \langle n|A|m\rangle \delta_{nm} = \sum_n f_n^*(\beta) f_n(\beta) \langle n|A|n\rangle \end{aligned} \tag{A9}$$

Here we have used the fact that an operator of the physical system does not act on the Hilbert space of the tilde system and vice versa. So, using the orthonormality of the states we can write

$$\begin{aligned} \langle n, \tilde{m}|A|n', \tilde{m}'\rangle &= \langle n|A|n'\rangle \langle \tilde{m}|\tilde{m'}\rangle = \delta_{mm'} \langle n|A|n'\rangle \\ \langle n, \tilde{m}|\tilde{A}|n', \tilde{m}'\rangle &= \langle n|\tilde{A}|n'\rangle \langle \tilde{m}|\tilde{m}'\rangle = \delta_{mm'} \langle n|\tilde{A}|n'\rangle \end{aligned} \tag{A10}$$

From (A3) and (A6) it is seen that

$$f_n^*(\beta) = f_n(\beta) = Z^{-1/2}(\beta) \exp(-\beta E_n/2) \tag{A11}$$

Now we can write the Fock space based on the vacuum $|0(\beta)\rangle$ which is defined by a Bogoliubov transformation as

$$|0(\beta)\rangle = U(\beta)|0, 0\rangle \tag{A12}$$

where

$$U(\beta) = \exp[-\theta(a\tilde{a} - a^\dagger \tilde{a}^\dagger)] \tag{A13}$$

where $\cosh(\theta) = 1/\sqrt{(1 - \exp(-\beta\omega))}$ and ω is the energy. Then the annihilation operators at finite temperature are defined as

$$a(\beta) = U^{-1}(\beta) a U(\beta), \qquad \tilde{a}(\beta) = U^{-1}(\beta) \tilde{a} U(\beta) \tag{A14}$$

such that

$$a(\beta)|0(\beta)\rangle = 0 \quad , \quad \tilde{a}(\beta)|0(\beta)\rangle = 0. \tag{A15}$$

Then Fock space is given as

$$\begin{aligned} &|0(\beta)\rangle, \quad a^\dagger(\beta)|0(\beta)\rangle, \quad \tilde{a}^\dagger(\beta)|0(\beta)\rangle, \ldots \\ &\ldots \qquad \tfrac{1}{\sqrt{n!}}\tfrac{1}{\sqrt{m!}}[a^\dagger(\beta)]^n[\tilde{a}^\dagger(\beta)]^m|0(\beta)\rangle \end{aligned} \tag{A16}$$

where n and m are the number of physical particles and tilde particles respectively. This therefore shows that we can introduce a temperature dependent vacuum state so that the statistical ensemble average of any operator can be identified with the expectation value of the operator in this state. This, however entails a doubling of the Hilbert space. The advantage, on the other hand , lies in the fact that the techniques of zero temperature field theory can now be carried over to the finite temperature case.

Here are the tilde conjugation rules [24]

$$\begin{aligned} &\widetilde{(AB)} = \tilde{A}\tilde{B}, \\ &\widetilde{(c_1 A + c_2 B)} = c_1^* \tilde{A} + c_2^* \tilde{B}, \\ &(\tilde{A^\dagger}) = (\tilde{A})^\dagger, \\ &\widetilde{|0(\beta)\rangle} = |0(\beta)\rangle \end{aligned} \tag{A17}$$

For example the tilde partner of the weak interaction Lagrangian

$$L_W = \frac{G_F}{\sqrt{2}} \bar{\nu}_\mu \Gamma_\alpha \mu \bar{e} \Gamma^\alpha \nu_e \tag{A18}$$

(in usual notations) is

$$\tilde{L}_W = \frac{G_F}{\sqrt{2}} \tilde{\bar{\nu}}_e \tilde{\Gamma}_\alpha \tilde{e} \tilde{\bar{\mu}} \tilde{\Gamma}^\alpha \tilde{\nu}_\mu, \tag{A19}$$

but not $\frac{G_F}{\sqrt{2}} \tilde{\bar{\nu}}_\mu \tilde{\Gamma}_\alpha \tilde{\mu} \tilde{\bar{e}} \tilde{\Gamma}^\alpha \tilde{\nu}_e$ which would give zero contribution to the amplitude of the process $\mu \to e \bar{\nu}_e \nu_\mu$ at tree level .

B. Green's functions

In TFD the thermal doublet for each field is defined as

$$\phi^{(a)}(x) \equiv \left\{ \begin{array}{c} \phi(x) \\ i\ {}^t\tilde{\phi}^\dagger \end{array} \right\} \equiv \left\{ \begin{array}{c} \phi_1(x) \\ \phi_2(x) \end{array} \right\} \tag{A20}$$

$$\psi^{(a)}(x) \equiv \left\{ \begin{array}{c} \psi(x) \\ i\ {}^t\tilde{\psi}^\dagger \end{array} \right\} \equiv \left\{ \begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array} \right\} \tag{A21}$$

where t means transposition with respect the spinor index, and $a(=1,2)$ specifies a component of the thermal doublet.The first componet is physical, and the second is fictitious.

The scalar free field may be written as [15]:

$$\begin{aligned} \phi(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{\sqrt{2\omega_p}} [a_p e^{-ipx} + a_p^\dagger e^{ipx}], \\ \tilde{\phi}(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{\sqrt{2\omega_p}} [\tilde{a}_p e^{ipx} + \tilde{a}_p^\dagger e^{-ipx}], \end{aligned} \tag{A22}$$

A Fermion field may be written as :

$$\begin{aligned} \psi(x) &= \sum_s \int d\vec{p} N_p [c_{p,s} u(p,s) e^{-ipx} + d_{p,s}^\dagger v(p,s) e^{ipx}], \\ \tilde{\psi}(x) &= \sum_s \int d\vec{p} N_p [\tilde{c}_{p,s} \tilde{u}(p,s) e^{ipx} + \tilde{d}_{p,s}^\dagger \tilde{v}(p,s) e^{-ipx}], \\ \bar{\psi}(x) &= \sum_s \int d\vec{p} N_p [c_{p,s}^\dagger \bar{u}(p,s) e^{ipx} + d_{p,s} \bar{v}(p,s) e^{-ipx}], \\ \bar{\tilde{\psi}}(x) &= \sum_s \int d\vec{p} N_p [\tilde{c}_{p,s}^\dagger \bar{\tilde{u}}(p,s) e^{-ipx} + \tilde{d}_{p,s} \bar{\tilde{v}}(p,s) e^{ipx}], \end{aligned} \tag{A23}$$

where $N_p = \sqrt{m/(2\pi)^3 E_p}$, $E_p = \sqrt{m^2 + (\vec{p})^2}$, $\tilde{u}(p,s) = u^*(p,s)$ etc.

The boson $\Delta_{ab}(k)$ and fermion $S_{ab}(k)$ propagators are $2 \otimes 2$ matrices. They are defined as:

$$\begin{aligned} i\Delta_{ab}(x-y) &= \langle 0(\beta)|T[\phi_a(x)\phi_b(y)]|0(\beta)\rangle = i\int \frac{dp^4}{(2\pi^4)} e^{-iq(x-y)} \Delta^0_{ab}(q) \\ iS_{ab}(x-y) &= \langle 0(\beta)|T[\psi_a(x)\bar{\psi}_b(y)]|0(\beta)\rangle = i\int \frac{dp^4}{(2\pi^4)} e^{-iq(x-y)} S^0_{ab}(q) \end{aligned} \tag{A24}$$

The free propagators may be presented as a sum of Feynman propagator, $\Delta^0_F(k)$, $\quad (S^0_F(k))$ and temperature dependent part, $\Delta^0_T(k)$, $\quad (S^0_T(k))$ as:

$$\Delta^0_{11}(k) = \Delta^0_F(k) + \Delta^0_T(k), \qquad \Delta^0_{22}(k) = -(\Delta^0_{11}(k))^*$$
$$\Delta^0_{12}(k) = -2\pi i\delta(k^2 - m^2)m_B(k), \qquad \Delta^0_{21}(k) = \Delta^0_{12}(k)$$
$$\Delta^0_F(k) = \frac{1}{k^2 - m^2 + i\varepsilon} \qquad \Delta^0_T(k) = -2\pi i n_B(k)\delta(k^2 - m^2),$$
$$S^0_{11}(k) = (\not k + m)(S^0_F(k) + S^0_T(k)), \qquad S^0_{22}(k) = (\not k + m)(S^0_F(k) + S^0_T(k))^* \tag{A25}$$
$$S_{12}(k)^0 = 2i\pi(\not k + m)\delta(k^2 - m^2)m_F(k) \qquad S_{21}(k)^0 = S_{12}(k)^0$$
$$S^0_F(k) = \frac{1}{k^2 - m^2 + i\varepsilon},$$
$$S^0_T(k) = 2i\pi(\not k + m)\delta(k^2 - m^2)\sin^2\theta_{k_0}.$$

Here the following notations are used

$$\sin^2\theta_{k_0} = \Theta(k_0)n_F(k) + \Theta(-k_0)\bar{n}_F(k),$$
$$m_F(k) = \frac{e^{x/2}[\Theta(k_0) - \Theta(-k_0)]}{e^x + 1} = \frac{sign(k_0)}{2}\sin 2\theta_{+k} \tag{A26}$$
$$m_B(k) = e^{\beta k_0/2}n_B(k) = \tfrac{1}{2}\sinh 2\theta_k, \qquad x = \beta(k_0 - \mu)$$

where Θ is the step function : $\Theta(k_0) = 1$ if $k_0 > 0$ and $\Theta(k_0) = 0$ otherwise and $sign(k_0)$ is the sign of k_0.

The full propagator satisfies the Dyson equation:

$$\Delta_{ab}(k) = \Delta^0_{ab}(k) + \sum_{cd}\Delta^0_{ac}(k)\Sigma^B_{cd}(k)\Delta_{db}(k)$$
$$S_{ab}(k) = S^0_{ab}(k) + \sum_{cd}S^0_{ac}(k)\Sigma^F_{cd}(k)S_{db}(k) \tag{A27}$$

where $\Sigma^F_{ab}(k)$ and $\Sigma^B_{ab}(k)$ are the proper self energies of a fermion and boson respectively. By introducing a complex function $\bar{\Sigma}_B = Re\bar{\Sigma}_B + iIm\bar{\Sigma}_B$ we can represent the general form of the self energy as follows:

$$\Sigma^B_{11}(k) = Re\bar{\Sigma}_B(k) + iIm\bar{\Sigma}_B(k)\cosh 2\theta_k, \qquad \Sigma^B_{22}(k) = -(\Sigma^B_{11}(k))^*$$
$$\Sigma^B_{12}(k) = -i\sinh 2\theta_k Im\bar{\Sigma}_B(k), \qquad \Sigma^B_{21}(k) = \Sigma^B_{12}(k) \tag{A28}$$

Similarly for the fermion self energy we have:

$$\Sigma^F_{11}(k) = Re\bar{\Sigma}_F(k) + i\cos 2\theta_k Im\bar{\Sigma}_F(k) \qquad \Sigma^F_{22}(k) = (\Sigma^F_{11}(k))^*$$
$$\Sigma^F_{12}(k) = -2iIm\bar{\Sigma}_F(k)m_F, \qquad \Sigma^F_{21}(k) = \Sigma^F_{12}(k) \tag{A29}$$

The full propagator has the following compact form in terms of $\bar{\Sigma}$:

$$\Delta_{11}(k) = \frac{\cosh^2\theta_k}{k^2 - m^2 - Re\bar{\Sigma}_B(k) - iIm\bar{\Sigma}_B(k) + i\varepsilon} - \frac{\sinh^2\theta_k}{k^2 - m^2 - Re\bar{\Sigma}_B(k) + iIm\bar{\Sigma}_B(k) - i\varepsilon}$$

$$\Delta_{12}(k) = m_B(k)\{\frac{1}{k^2 - m^2 - Re\bar{\Sigma}_B(k) - iIm\bar{\Sigma}_B(k) + i\varepsilon} - \frac{1}{k^2 - m^2 - Re\bar{\Sigma}_B(k) + iIm\bar{\Sigma}_B(k) - i\varepsilon}\}$$

$$\Delta_{22}(k) = -(\Delta_{11}(k))^*, \qquad \Delta_{21}(k) = \Delta_{12}(k) \tag{A30}$$

$$S_{11}(k) = \frac{\cos^2\theta_{k_0}}{\not{p} - m - Re\bar{\Sigma}_F(k) - iIm\bar{\Sigma}_F(k) + i\varepsilon} + \frac{\sin^2\theta_{k_0}}{\not{p} - m - Re\bar{\Sigma}_F(k) + iIm\bar{\Sigma}_F(k) - i\varepsilon}$$

$$S_{12}(k) = -m_F(k)\{\frac{1}{\not{p} - m - Re\bar{\Sigma}_F(k) - iIm\bar{\Sigma}_F(k) + i\varepsilon} + \frac{1}{\not{p} - m - Re\bar{\Sigma}_F(k) + iIm\bar{\Sigma}_F(k) - i\varepsilon}\}$$

$$S_{21}(k) = S_{12}(k) \qquad S_{22}(k) = (S_{11}(k))^* \quad , \tag{A31}$$

where $\cosh^2\theta_k = 1 + n_B$, $\sinh^2\theta_k = n_B(k)$, $\cos^2\theta_{k_0} = 1 - \sin^2\theta_{k_0}$ and $\sin^2\theta_{k_0}$ was defined in Eq. (A26). The spectral representation for propagators and their matrix form may be found in ref. [15].

REFERENCES

[1] Ashok Das, Lectures on finite temperature field theory University of Rochester, 1996; F.C. Khanna and A.E. Santana, Topics in Thermo Field Dynamics, Lectures given at the second International schooll on field theory and gravitation. Vitoria, Brazil, 1999.

[2] Y. Takahashi and H. Umezawa, Int. Journ. Mod. Phys. 10, 1755 (1996)

[3] T. Matsubara, Progr. Theor. Phys., **14**, 351 (1955).

[4] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).

[5] J. Schwinger, J. Math. Phys. **2**, 407 (1961);
L.V. Keldish, Sov. Phys. JETP, **20**, 1018 (1964).

[6] A. Niegawa, Phys.Rev.D **57**, 1379 (1998).

[7] R. Feymnan, Rev. Mod. Phys. **20**, 367 (1948).

[8] F. Dyson, Phys. Rev, **75**, 1736 (1949).

[9] F. Mandl and G. Shaw, Quantum field theory. (A Wiley - Interscience 1986).

[10] F. Cutkosky, Jour. Math. Phys. **1**, 429 (1960)

[11] R. Kobes and G. Semenoff, Nucl. Phys. **B260**, 714 (1985);
R. Kobes and G. Semenoff, Nucl. Phys. **B272**, 329 (1986).

[12] Y.Fujimoto, M. Morikawa and M. Sasaki, Phys. Rev. D **33**, 590 (1986)

[13] W. Keil, Phys. Rev. D **40**, 1176 (1989).

[14] H. A. Weldon, Phys. Rev. D **28**, 2007 (1983).

[15] K. Saito, T. Maruyama and K. Soutome, Phys. Rev.C **40**, 407 (1989).

[16] M. Matsumoto, I. Ojima and H. Umezawa, Ann. Phys. (N.Y.) **152**, 348 (1984).

[17] P. Ramond, Field theory of modern primer. (Benjamin/cumming publishing company) (1981).

[18] S.J.Hardy and M.H. Thomas, Neutrino - electron processes in a strongly magnetized thermal field, astro-ph/0008473, and references therein.

[19] V. G. Bezchastnov and P. Haensel Phys. Rev. D **54** , 3706 (1996)

[20] R. Machleidt, Microscopic calculation of in - medium proton - proton cross sections. nucl-th/9308016

[21] A.M. Rakhimov, U.T. Yakhshev and F.C. Khanna . Phys.Rev.C **61**, 024907(2000)

[22] Yi - Jun Zhang, Song Gao and Ru - Keng Su Phys. Rev. C **56**, 3336 (1997) and references there in.

[23] W. Zhao and F.C. Khanna, to be published.

[24] H. Umezawa, H. Matsumoto and H. Tachiki, Thermo Field Dynamics and condensed states (North- Holland, Amsterdam, 1982);
H. Umezawa, Advanced field theory, Micro, Macro and Thermal Physics, (AIP 1994).

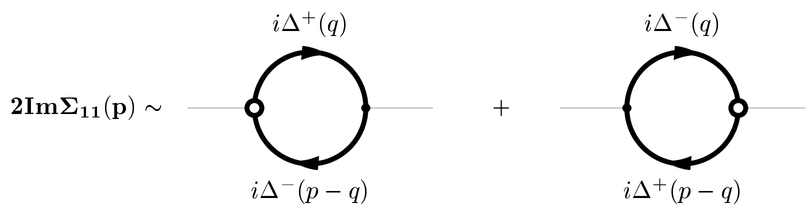

Fig. 1. Generalized Cutcosky rules for the Imaganary part of boson self energy. See ref. [11] for the details.

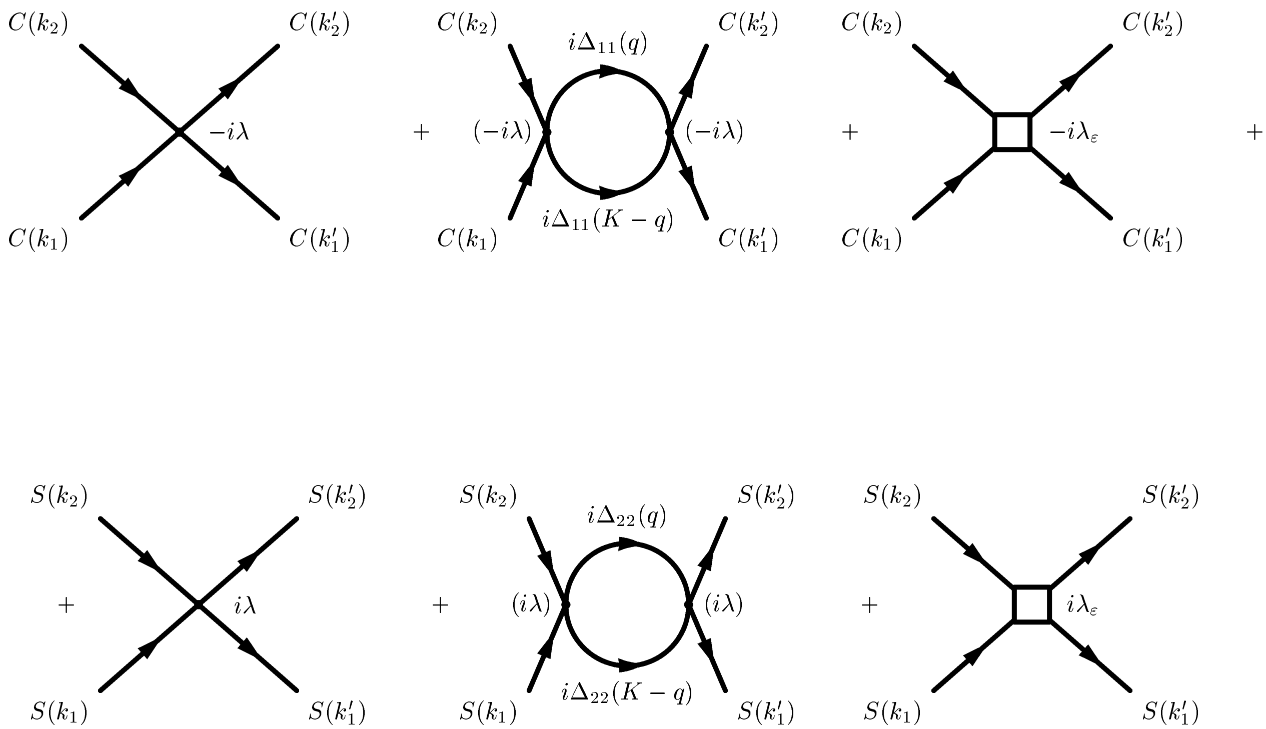

Fig. 2 Feynman diagrams for boson - boson scattering amplitude up λ^2 in $\lambda\phi^4(x)/4!$ model at finite temperature. Here the following notations are used : $C(k) \equiv \cosh\theta_k$, $S(k) \equiv \sinh\theta_k$, $K = k_1 + k_2 = {k_1}' + {k_2}'$ and $\lambda_\varepsilon = \lambda^2 N_\varepsilon/16\pi^2$